

Derivation of growth equations

The different forms of the Richards, Hossfeld IV and Korf models listed in Table 2 of the paper were derived using the GADA technique described by Cieszewski & Bailey (2000). Below are details of these derivations.

Richards model

Base form:

$$(S1) \quad y = a(1 - e^{-bt})^c$$

Anamorphic form:

The GADA technique uses a theoretical variable x which quantifies how the growth curve varies in shape or level across different sites. For an anamorphic model, the asymptote parameter, a , is replaced by x , i.e.,

$$(S2) \quad y = x(1 - e^{-bt})^c$$

This equation is solved for x at point (t_0, y_0) giving,

$$x = y_0 / (1 - e^{-bt_0})^c$$

Substituting this into Equation (S2) produces the required anamorphic equation,

$$(S3) \quad y = y_0 \left(\frac{1 - e^{-bt}}{1 - e^{-bt_0}} \right)^c$$

Inverting this equation produces the following *SI* equation where t_0 is the base age,

$$y_0 = y \left(\frac{1 - e^{-bt}}{1 - e^{-bt_0}} \right)^{-c}$$

Common-asymptote form:

For the common-asymptote model, the time-scale parameter, b , in the base model is replaced by x ,

$$(S4) \quad y = a(1 - e^{-xt})^c$$

Solving for x at point (t_0, y_0) ,

$$x = -\ln(1 - (y_0/a)^{1/c}) / t_0$$

Substituting this into Equation (S4) produces the required common-asymptote equation,

$$(S5) \quad y = a \left(1 - \left(1 - \left(\frac{y_0}{a} \right)^{1/c} \right)^{t/t_0} \right)^c$$

Inverting this equation produces the following *SI* equation,

$$y_0 = a \left(1 - \left(1 - (y/a)^{1/c} \right)^{t_0/t} \right)^c$$

Hossfeld IV model

Base form:

$$(S6) \quad y = t^c / (b + t^c / a)$$

Anamorphic form:

To derive the anamorphic form, replace the parameter, a , with x ,

$$(S7) \quad y = t^c / (b + t^c / x)$$

Solve for x at point (t_0, y_0) ,

$$x = t_0^c / (t_0^c / y_0 - b)$$

Substituting this into Equation (S7) produces the required anamorphic equation,

$$(S8) \quad y = t^c / (b + t^c (1/y_0 - b/t_0^c))$$

Inverting this equation produces the following *SI* equation,

$$y_0 = 1 / (1/y + b(1/t_0^c - 1/t^c))$$

Common-asymptote form:

To derive the common-asymptote form, replace the parameter, b , with x ,

$$(S9) \quad y = t^c / (x + t^c / a)$$

Solve for x at point (t_0, y_0) ,

$$x = t_0^c (1/y_0 - 1/a)$$

Substituting this into Equation (S9) produces the required common-asymptote equation,

$$(S10) \quad y = t^c / (t_0^c / y_0 + (t^c - t_0^c) / a)$$

Inverting this equation produces the following *SI* equation,

$$y_0 = t_0^c / (t^c / y - (t^c - t_0^c) / a)$$

Korf model

Base form:

$$(S11) \quad y = ae^{-bt^{-c}}$$

Anamorphic form:

To derive the anamorphic form, replace the parameter, a , with x ,

$$(S12) \quad y = xe^{-bt^{-c}}$$

Solve for x at point (t_0, y_0) ,

$$x = y_0 / e^{-bt_0^{-c}}$$

Substituting this into Equation (S12) produces the required anamorphic equation,

$$(S13) \quad y = y_0 \frac{e^{-bt^{-c}}}{e^{-bt_0^{-c}}}$$

Inverting this equation produces the following *SI* equation,

$$y_0 = y \frac{e^{-bt_0^{-c}}}{e^{-bt^{-c}}}$$

Common-asymptote form:

To derive the common-asymptote form, replace the parameter, b , with x ,

$$(S14) \quad y = xe^{-xt^{-c}}$$

Solve for x at point (t_0, y_0) ,

$$x = -\ln(y_0/a)/t_0^{-c}$$

Substituting this into Equation (S14) produces the required common-asymptote equation,

$$(S15) \quad y = a \left(\frac{y_0}{a} \right)^{(t/t_0)^{-c}}$$

Inverting this equation produces the following *SI* equation,

$$y_0 = \left(\frac{y}{a} \right)^{-(t/t_0)^{-c}}$$

GADA polymorphic form:

To derive the GADA polymorphic form, we start by assuming that x is proportional to the exponential of the asymptote, a , and inversely related to the time scale parameter, b ,

$$(S16) \quad y = e^x e^{-(b/x)t^{-c}}$$

Taking logs and multiplying both sides of the equation by x produces the following quadratic equation,

$$x^2 - x \ln y - bt^{-c} = 0$$

Solving for x at point (t_0, y_0) produces,

$$x = \frac{\ln y_0 + \sqrt{(\ln y_0)^2 + 4bt_0^{-c}}}{2}$$

Substituting this into Equation (S16) produces the required polymorphic equation,

$$(S17) \quad y = e^{R_0/2 - 2b/(R_0 t^c)}, \text{ where, } R_0 = \ln y_0 + \sqrt{(\ln y_0)^2 + 4b/t_0^c}$$

To produce a *SI* equation for this model, first take logs of (S17) and multiply by R_0 to produce the quadratic equation,

$$R_0^2/2 - R_0 \ln y - 2b/t^c = 0$$

Solving for R_0 produces,

$$R_0 = \ln y + \sqrt{(\ln y)^2 + 4b/t^c}$$

Which leads to,

$$\sqrt{(\ln y_0)^2 + 4b/t_0^c} = R - \ln y_0, \text{ where, } R = \ln y + \sqrt{(\ln y)^2 + 4b/t^c}$$

Squaring both sides produces,

$$(\ln y_0)^2 + 4b/t_0^c = R^2 - 2R \ln y_0 + (\ln y_0)^2$$

Which simplifies to the required *SI* equation,

$$y_0 = e^{(4b/t_0^c - R^2)/(-2R)}$$

Reference

Cieszewski, C.J. and Bailey, R.L., 2000. Generalized algebraic difference approach: theory based derivation of dynamic site equations with polymorphism and variable asymptotes. *Forest Science* 46, 116-126.